Semiclassical model for quantum dissipation

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We consider the simple idea of coupling a quantum system to a dissipative classical one through well-defined quantities of the former. The dynamical evolution is described via Ehrenfest's theorem. This model is able to mimic a dissipative temporal evolution, without violation of any quantal rule.

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I. INTRODUCTION

Sustained effort has over the years been devoted to the goal of trying to understand quantum models by recourse to systems described by a few classical variables [1]. These systems can be motivated, in the $\hbar \to 0$ limit, either by the effective potential approaches [2,3] or by semiclassical treatments (WKB, for instance). In this vein, systems characterized by the coexistence of both classical and quantum degrees of freedom have been recently employed. For example, Bonilla and Guinea have in such a fashion described measurement processes [4] and Pattanayak and Schieve have studied quantum chaos by recourse to an appropriate, effective classical Hamiltonian [5].

The interplay between quantal and classical variables has acquired special relevance in connection with the concept of quantum friction. Different attempts to quantify dissipative forces have received renewed attention, as experimental evidence has been accumulating with regard to the presence of dissipation phenomena in several microscopic processes [7]. In spite of the fact that several techniques have been employed, a unanimously accepted prescription for quantifying dissipating systems has not yet been devised. There are two major approaches for the quantum mechanical handling, as a one-body problem [1,8–13], of the damped motion of a particle: (a) the historical, Kanai [8] method, in which an explicitly time-dependent Hamiltonian is used, and (b) introducing a "friction" potential, which relies upon specially selected expectation values [8,9]. The main criticisms that have been made with respect to these two kinds of Hamiltonians allude to the fact that there exists an apparent violation of Heisenberg's uncertainty principle and that, additionally, the proposed Hamiltonians do not coincide in general with the energy operator [11,14,15].

It is of importance to point out that one faces an easily solvable set of equations for the description of the time evolution of expectation values of, say, q operators in which one may be interested, *if these operators close a partial Lie algebra with respect to the Hamiltonian $\hat{H}$ of the system*, i.e., if we have a set of relations of the type

$$[\,\hat{H}(t), \hat{O}_i\,] = i\hbar \sum_{j=1}^{q} g_{ji}(t)\hat{O}_j, \qquad i = 1, 2, \ldots, q, \quad (1.1)$$

where the g_{ji} are the elements of a $q \times q$ matrix G. If such is the case we obtain, from the generalized Ehrenfest theorem,

$$\frac{d\langle \hat{O}_i \rangle}{dt} = \frac{i}{\hbar}\langle [\,\hat{H}, \hat{O}_i\,] \rangle, \qquad i = 1, 2, \ldots, q, \quad (1.2)$$

a linear set of first-order differential equations

$$\frac{d\langle \hat{O}_i \rangle}{dt} = -\sum_{j=1}^{q} g_{ji}(t)\langle \hat{O}_j \rangle, \qquad i = 1, 2, \ldots, q, \quad (1.3)$$

for the temporal evolution of the expectation values. Many instances of physical interest are encompassed within these particular circumstances [16,17].

In this paper we shall study the interaction between a quantum system and a classical one and assume that the classical degrees of freedom obey the (deterministic) equations of motion usually employed in describing classical friction. The resulting type of interaction, as we show below, is able to mimic a dissipative quantum behavior without violating any quantum rules. The paper is organized as follows. In Sec. II we present the model. Sec. III is devoted to illustrative examples. Finally, some conclusions are drawn in Sec. IV.

II. MODEL

We consider the interaction between a quantum system and a classical one described by a Hamiltonian of the form

$$\hat{H} = \hat{H}_q + H_{\rm cl} + \hat{H}^q_{\rm cl}, \quad (2.1)$$

where $\hat{H}_q$ and $H_{\rm cl}$ stand for quantal and classical Hamil-

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tonians, respectively, and $\hat{H}^q_{\rm cl}$ is an interaction potential. Our main goal is that of describing dissipative behavior while avoiding any violation of the Heisenberg principle. Previously proposed Hamiltonians do not coincide with the energy operator [8,10,14,15]. This is not the case here.

The dynamical equations for the quantal (relevant) variables are the canonical ones [Eqs. (1.2)], which in turn will also depend on the classical ones through the g_{ji} elements of the matrix G of Eqs. (1.3). Our central idea is that of assigning to the classical variables that type of evolution prescribed by the classical treatment of dissipative systems [18]. The energy is taken here to coincide with the quantum expectation value of the Hamiltonian (2.1) and in turn generates the temporal evolution of all the classical variables in the frictionless case. Consequently, the classical equations of motion to be used here are well-defined ones. If we take the classical variables to be a position s and a momentum p_s, we set

$$\frac{ds}{dt}=\frac{\partial\langle\hat{H}\rangle}{\partial p_s}, \tag{2.2a}$$

$$\frac{dp_s}{dt}=-\left(\frac{\partial\langle\hat{H}\rangle}{\partial s}+\eta p_s\right). \tag{2.2b}$$

The parameter $\eta>0$ is a dissipative one and plays a prominent role in the present considerations. Of course, the second term on the right-hand-side of (2.2b) appears in the *ad hoc* fashion usually employed in describing classical friction [18]. Through this parameter η, the classical variable is coupled to an appropriate reservoir. Energy is dissipated into this reservoir. Consequently, the last term on the right-hand-side of (2.1) allows one to think of "quantum dissipation," albeit via an indirect route: the quantum system (for example, a degree of freedom of a system) interacts with a classical one (the rest of the system, whose behavior may be considered as classical), which in turn is coupled to the reservoir (the environment) [see below Eq. (3.50)]. The central idea of the present work is that of discussing quantum friction using this indirect route, which allows for a dynamical description [see below Eqs. (3.3) and (3.28)] in which no quantum rules are violated.

We consider in this work a peculiar space, which shall be referred to as the "u space," in order to pursue our investigations. The set of equations derived from Eqs. (1.3) (for variables belonging to the quantal system) and from (2.2) (for the classical variables) configure an autonomous set of first-order coupled differential equations of the form

$$\frac{d\vec{u}}{dt}=\vec{F}(\vec{u}), \tag{2.3}$$

where $\vec{u}$ is an appropriate, generalized variable (a "vector" with both classical and quantum components). If one considers an arbitrary volume element V_S enclosed by a surface S in the concomitant u space, the dissipative η term induces a contraction of V_S [19] [the divergence of $\vec{F}$ is easily seen to be $-\eta$ since the matrix G in the set of equations (1.3) is traceless, on account of the canonical nature of Eqs. (1.2) and (1.3)]. One is thus led to

$$\frac{dV_S(t)}{dt}=-\eta V_S(t), \tag{2.4}$$

which entails that our system is a dissipative one [20]. If the classical Hamiltonian adopts the general appearance

$$H_{\rm cl}=\frac{1}{2M}p_s^2+V(s), \tag{2.5}$$

one easily ascertains that the temporal evolution for the total energy $\langle\hat{H}\rangle$ is given by

$$\frac{d\langle\hat{H}\rangle}{dt}=-\frac{\eta}{M}p_s^2, \tag{2.6}$$

whose significance is to be appreciated in the light of Eq. (2.4).

The set of relationships given by Eq. (1.1) includes the basic conmutator $[\hat{x},\hat{p}]=i\hbar$, directly connected with the uncertainty principle. This conmutation relation (and related ones) is trivially conserved for all time (the quantal evolution is the canonical one), so that one is able to avoid any quantum pitfall [11,14,15]. In the following section we give two examples of the dissipative machinery envisioned in writing down the equations of the present section.

III. EXAMPLES

A. Quantum harmonic oscillator coupled to a classical harmonic oscillator

The simplest Hamiltonian we can think of is that of two coupled harmonic oscillators of frequencies ω_0 and ω, respectively,

$$\hat{H}=\frac{1}{2}\left(\frac{1}{m}\hat{p}_q^2+m\omega_0^2\hat{x}_q^2+\frac{1}{M}p_{\rm cl}^2+M\omega^2x_{\rm cl}^2\right)+\gamma x_{\rm cl}\hat{x}_q, \tag{3.1}$$

where $\hat{p}_q^2$, $\hat{x}_q^2$, and $\hat{x}_q$ are quantum operators and $p_{\rm cl}$ and $x_{\rm cl}$ classical variables. The condition $(\gamma^2/mM)\le\omega_0^2\omega^2$ guarantees elimination of divergent components of the pertinent solutions [see Eq. (3.9)].

First of all, we introduce the usual dimensionless operators and classical variables ($\hbar=1$)

$$\hat{x}=(m\omega_0)^{1/2}\hat{x}_q, \tag{3.2a}$$

$$\hat{p}=\frac{\hat{p}_q}{(m\omega_0)^{1/2}}, \tag{3.2b}$$

$$s=(M\omega)^{1/2}x_{\rm cl}, \tag{3.2c}$$

$$p_s=\frac{p_{\rm cl}}{(M\omega)^{1/2}}. \tag{3.2d}$$

A partial Lie algebra [cf. Eq. (1.1)] ensues if we choose as relevant operators those belonging to the set $\{\hat{1},\ \hat{x},\ \hat{p},\ \hat{x}^2,\ \hat{p}^2,\ \hat{L}=\hat{x}\hat{p}+\hat{p}\hat{x}\}$, where $\hat{1}$ is the unity operator and

$\hat{L}$ is referred to as the correlation operator. The sets $\{\hat{1}, \hat{p}, \hat{x}\}$ and $\{\hat{x}^2, \hat{p}^2, \hat{L}\}$ correspond to the Heisenberg and the S(1,1) groups, respectively. Applying the generalized Ehrenfest theorem to the expectation values of the above mentioned operators we immediately arrive at the system of coupled differential equations

$$\frac{d\langle\hat{x}\rangle}{d\tau} = \langle\hat{p}\rangle, \tag{3.3a}$$

$$\frac{d\langle\hat{p}\rangle}{d\tau} = -(\langle\hat{x}\rangle + \chi s), \tag{3.3b}$$

$$\frac{d\langle\hat{x}^2\rangle}{d\tau} = \langle\hat{L}\rangle, \tag{3.3c}$$

$$\frac{d\langle\hat{p}^2\rangle}{d\tau} = -(\langle\hat{L}\rangle + 2\chi s\langle\hat{p}\rangle), \tag{3.3d}$$

$$\frac{d\langle\hat{L}\rangle}{d\tau} = 2(\langle\hat{p}^2\rangle - \langle\hat{x}^2\rangle - \chi s\langle\hat{x}\rangle), \tag{3.3e}$$

where $\tau = \omega_0 t$ and $\chi = \gamma/(mM\omega\omega_0^3)^{1/2}$ are useful dimensionless quantities. Equations (3.3) are the dynamical equations for the quantum relevant variables, which depend on the classical coordinate s. In particular, for $\gamma = 0$ and $\omega_0 = \omega(0)\exp(-kt)$ Eqs. (3.3) coincide with those employed in reference to the Kanai Hamiltonian [8]. For the classical variables we obtain

$$\frac{ds}{d\tau} = \Omega p_s, \tag{3.4a}$$

$$\frac{dp_s}{d\tau} = -(\chi\langle\hat{x}\rangle + \Omega s + \delta p_s), \tag{3.4b}$$

where $\Omega = \omega/\omega_0$ and $\delta = \eta/\omega_0$. Notice that Eqs. (3.3a), (3.3b), and (3.4) configure an autonomous set of differential equations by themselves. With Eqs. (3.3) and (3.4) the general form for the temporal evolution of the expectation values of $\hat{x}$ and $\hat{p}$ can be cast in the form

$$\langle\hat{x}\rangle(\tau) = \sum_{k=1}^{4} A_k \exp(r_k\tau), \tag{3.5a}$$

$$\langle\hat{p}\rangle(\tau) = \sum_{k=1}^{4} r_k A_k \exp(r_k\tau), \tag{3.5b}$$

while for the classical variables s and p_s we obtain

$$s(\tau) = -\frac{1}{\chi}\left\{\sum_{k=1}^{4}(r_k^2+1)A_k\exp(r_k\tau)\right\}, \tag{3.6a}$$

$$p_s(\tau) = -\frac{1}{\Omega\chi}\left\{\sum_{k=1}^{4}(r_k^2+1)r_k A_k\exp(r_k\tau)\right\}. \tag{3.6b}$$

In these equations we have employed the abbreviations

$$A_k = B_k\{\langle\hat{x}\rangle(0)r_k[r_k(\delta+r_k)-\Omega^2] + s(0)\chi(\delta+r_k) - \langle\hat{p}\rangle(0)[r_k(\delta+r_k)+\Omega^2] + p_s(0)\chi\Omega\}, \tag{3.7}$$

where $B_k = -[4r_k^3 + 3r_k^2 + 2(\Omega^2+1)r_k + \delta]^{-1}$ and the r_k's are the roots of the fourth degree equation

$$r^4 + \delta r^3 + (\Omega^2+1)r^2 + \delta r + \Omega(\Omega - \chi^2) = 0. \tag{3.8}$$

In general, the roots of Eq. (3.8) can be cataloged in the following fashion: (i) two pairs of complex conjugate roots, (ii) two real roots together with a pair of complex conjugate ones, and (iii) four real roots. The real part of the complex conjugate roots does not vanish, except when $\chi = 0$, which is not the case we are interested in here. Negative real roots and negative real parts of the complex roots are obtained for

$$\Omega \geq \chi^2. \tag{3.9}$$

If instead $\Omega < \chi^2$, a nondissipative dynamics results from the evolution equations. Notice that the condition given by Eq. (3.9) allows for the elimination of divergent components from the pertinent solutions, independently of the presence or lack thereof of a dissipative factor. The condition (3.9) asserts that the coupling between the systems must be weak enough for our indirect quantum dissipation mechanism to work.

The analytical solution of Eq. (3.8) (for the general case) does not provide one with much useful insight. In order to obtain results amenable to useful interpretation, we have considered the special instance for which the particular relationship

$$\delta^2 = 4(\Omega^2 - 1) \tag{3.10}$$

holds. In this case, we can easily transform Eq. (3.8) into a quadratic one by introducing the variable

$$v = r + \frac{\delta}{4}. \tag{3.11}$$

After a careful analysis of the concomitant quadratic equation (see Appendix A) it is possible to ascertain that three regimes can be defined. They are illustrated in Fig. 1, where we plot α^2 vs $(\delta/2)^2$ with

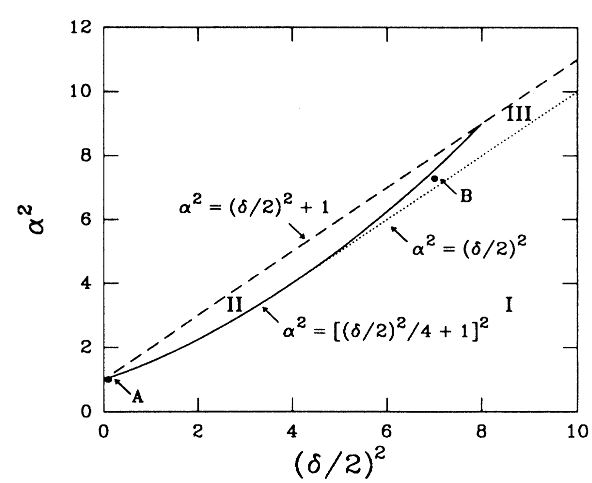

FIG. 1. Domains of the different regimes for the roots of Eq. (3.8). Region I, two pairs of complex conjugate roots; region II, a pair of complex conjugate and two real roots; region III, four real roots. The full line represents the curve $\alpha^2 = [(\delta/2)^2/4 + 1]^2$, the dashed line represents the curve $\alpha^2 = (\delta/2)^2 + 1$, and the dotted line represents the curve $\alpha^2 = (\delta/2)^2$. Points A and B are selected in order to draw Figs. 2 and 3, respectively.

$$\alpha = \chi\Omega^{1/2} = \frac{\gamma}{(mM\omega_0^4)^{1/2}}. \qquad (3.12)$$

The domains of each of the above mentioned cases (i), (ii), and (iii) correspond, respectively, to the regions labeled I, II, and III. More details are given in the pertinent figure caption. From the results of Appendix A one learns that the quantities $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, s, and p_s vanish in the limit $\tau \to \infty$, except for the particular case $\Omega = \chi^2$, for which the final values $\langle\hat{x}\rangle_f$ and s_f of, respectively, $\langle\hat{x}\rangle$ and s are related in the fashion

$$\begin{aligned}\langle\hat{x}\rangle_f &= -\chi s_f \\ &= \frac{1}{\delta}(\Omega^2\langle\hat{p}\rangle(0) - \chi\delta s(0) - \chi\Omega p_s(0)). \qquad (3.13)\end{aligned}$$

Only in the case of the largest [cf. Eq. (3.9)] acceptable coupling ($\Omega = \chi^2$) is the quantum system perturbed in such a fashion that, in the limit $\tau \to \infty$, the two oscillators remain coupled. The final expectation value of $\hat{x}$ can be obtained by looking at the value of the classical coordinate s. The expectation values $\langle\hat{x}^2\rangle$, $\langle\hat{p}^2\rangle$, and $\langle\hat{L}\rangle$ are to be evaluated according to

$$\begin{aligned}\langle\hat{x}^2\rangle(\tau) =& \frac{1}{2}[\langle\hat{K}_{11}\rangle(0) - \langle\hat{K}_{22}\rangle(0)]\cos(2\tau) + \langle\hat{K}_{12}\rangle(0)\sin(2\tau) \\ &+ \frac{1}{2}[\langle\hat{K}_{11}\rangle(0) + \langle\hat{K}_{22}\rangle(0)] \\ &+ \sum_{k=1}^{4}\sum_{j=1}^{4} A_k A_j \exp[(r_k + r_j)\tau], \qquad (3.14a)\end{aligned}$$

$$\begin{aligned}\langle\hat{p}^2\rangle(\tau) =& \frac{1}{2}[\langle\hat{K}_{22}\rangle(0) - \langle\hat{K}_{11}\rangle(0)]\cos(2\tau) \\ &- \langle\hat{K}_{12}\rangle(0)\sin(2\tau) + \frac{1}{2}[\langle\hat{K}_{11}\rangle(0) + \langle\hat{K}_{22}\rangle(0)] \\ &+ \sum_{k=1}^{4}\sum_{j=1}^{4} A_k A_j r_k r_j \exp[(r_k + r_j)\tau], \qquad (3.14b)\end{aligned}$$

$$\begin{aligned}\langle\hat{L}\rangle(\tau) =& 2[\langle\hat{K}_{12}\rangle(0)]\cos(2\tau) \\ &+ [\langle\hat{K}_{22}\rangle(0) - \langle\hat{K}_{11}\rangle(0)]\sin(2\tau) \\ &+ \sum_{k=1}^{4}\sum_{j=1}^{4} A_k A_j (r_k + r_j)\exp[(r_k + r_j)\tau], \qquad (3.14c)\end{aligned}$$

where

$$\langle\hat{K}_{11}\rangle = (\Delta x)^2 = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2, \qquad (3.15a)$$
$$\langle\hat{K}_{22}\rangle = (\Delta p)^2 = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2, \qquad (3.15b)$$
$$\langle\hat{K}_{12}\rangle = \frac{1}{2}\langle\hat{L}\rangle - \langle\hat{x}\rangle\langle\hat{p}\rangle, \qquad (3.15c)$$

are the customary quantum correlations. These expectation values have, for $\tau \to \infty$, an oscillatory behavior, and the pertinent trajectory in u space configures a closed orbit (limit cycle) determined by the invariants of the motion [21–23]

$$I_1 = \langle\hat{K}_{11}\rangle + \langle\hat{K}_{22}\rangle, \qquad (3.16a)$$
$$I_2 = \langle\hat{K}_{11}\rangle\langle\hat{K}_{22}\rangle - \langle\hat{K}_{12}\rangle^2. \qquad (3.16b)$$

More precisely, for $\tau \to \infty$ the system is constrained to the curve (in u space) given by the set of six equations

$$I_1 = \langle\hat{K}_{11}\rangle + \langle\hat{K}_{22}\rangle, \qquad (3.17a)$$
$$I_2 = \langle\hat{K}_{11}\rangle\langle\hat{K}_{22}\rangle - \langle\hat{K}_{12}\rangle^2, \qquad (3.17b)$$
$$\langle\hat{p}\rangle = 0, \qquad (3.17c)$$
$$p_s = 0, \qquad (3.17d)$$

and either $\langle\hat{x}\rangle = s = 0$ (for $\Omega \neq \chi^2$) or (for $\Omega = \chi^2$) $\langle\hat{x}\rangle = \langle\hat{x}\rangle_f$ and $s = s_f$, where $\langle\hat{x}\rangle_f$ and s_f are given by Eq. (3.13) (the fifth and sixth equations).

Heisenberg's uncertainty principle reads here

$$\begin{aligned}(\Delta x)^2(\Delta p)^2(\tau) =& \langle\hat{K}_{11}\rangle(0)\langle\hat{K}_{22}\rangle(0) - [\langle\hat{K}_{12}\rangle(0)]^2 \\ &+ \{\tfrac{1}{2}[-\langle\hat{K}_{11}\rangle(0) + \langle\hat{K}_{22}\rangle(0)]\sin(2\tau) \\ &+ \langle\hat{K}_{12}\rangle(0)\cos(2\tau)\}^2. \qquad (3.18)\end{aligned}$$

Notice that $(\Delta x)^2(\Delta p)^2$ is δ independent. Thus, if Heisenberg's uncertainty principle is satisfied for $\tau = 0$, one always has [24]

$$(\Delta x)^2(\Delta p)^2(\tau) \geq \langle\hat{K}_{11}\rangle(0)\langle\hat{K}_{22}\rangle(0) - [\langle\hat{K}_{12}\rangle(0)]^2 \geq \frac{1}{4}, \qquad (3.19)$$

so that no difficulties of the type encountered by other authors [11,14,15] arise.

The expression for the quantal energy, i.e., the mean value of

$$\hat{H}_q = \frac{1}{2}\omega_0(\hat{x}^2 + \hat{p}^2), \qquad (3.20)$$

is deduced from Eqs. (3.14a) and (3.14b). One finds

$$\begin{aligned}\langle\hat{H}_q\rangle(\tau) = \frac{1}{2}\omega_0\Bigg\{&\langle\hat{K}_{11}\rangle(0) + \langle\hat{K}_{22}\rangle(0) \\ &+ \sum_{k=1}^{4}\sum_{j=1}^{4} A_k A_j (r_k r_j + 1)\exp[(r_k + r_j)\tau]\Bigg\}. \qquad (3.21)\end{aligned}$$

It is easy to prove that the relationship $\langle\hat{H}_q\rangle \geq \omega_0/2$ is satisfied for all time.

If $\Omega > \chi^2$ the quantal and classical oscillators are seen to decouple for $\tau \to \infty$. We obtain, for the final value $\langle\hat{H}_q\rangle_f = \langle\hat{H}_q\rangle(\infty)$,

$$\langle\hat{H}_q\rangle_f = \frac{1}{2}\omega_0[\langle\hat{K}_{11}\rangle(0) + \langle\hat{K}_{22}\rangle(0)], \qquad (3.22)$$

which is the minimum value of $\langle\hat{H}_q\rangle$ (and also of $\langle\hat{H}\rangle$) for the given initial conditions as represented by the invariants (3.16), so that dissipation clearly ensues, and one finds the value

$$\begin{aligned}\Delta E_q &= \langle\hat{H}_q\rangle_f - \langle\hat{H}_q\rangle(0) \\ &= -\frac{1}{2}\omega_0\{[\langle\hat{x}\rangle(0)]^2 + [\langle\hat{p}\rangle(0)]^2)\} \leq 0 \qquad (3.23)\end{aligned}$$

for the energy variation ΔE_q.

If $\Omega = \chi^2$, our two systems remain coupled for all τ and there is no way to differentiate a "quantal" energy from a "classical" one. Here the Hamiltonian (3.1) may be written in the form

$$\hat{H} = \frac{1}{2}\omega_0[(\hat{x} + \chi s)^2 + \hat{p}^2 + \Omega p_s^2]. \quad (3.24)$$

Thus, in the limit $\tau \to \infty$, for which the classical variable s (and also the potential interaction) does not vanish, we obtain a "final" Hamiltonian

$$\hat{H}_f = \frac{1}{2}\omega_0[(\hat{x} - \langle\hat{x}\rangle_f)^2 + \hat{p}^2], \quad (3.25)$$

so that the Hamiltonian (3.24) describes a quantum oscillator with a time-dependent equilibrium position that for $\tau \to \infty$ is equal to $\langle\hat{x}\rangle_f$. The quantity $\langle\hat{H}_f\rangle$ is the minimum value of $\langle\hat{H}\rangle$ for the given initial conditions represented by the invariants (3.16) and in this case we obtain, for the energy variation, an expression similar to Eq. (3.23), i.e., one has

$$\Delta E_q = \langle\hat{H}_f\rangle - \langle\hat{H}_q\rangle(0) = -\frac{1}{2}\omega_0\{[\langle\hat{x}\rangle(0)]^2 + [\langle\hat{p}\rangle(0)]^2\} \le 0. \quad (3.26)$$

We gather from Eqs. (3.14), (3.18), and (3.22) that the quantum behavior is determined by the correlations (3.15) at $\tau = 0$. In the classical limit, the correlations vanish and for $\tau \to \infty$ so does the energy (3.22). The temporal evolution of the expectation values of the relevant operators [given by Eqs. (3.5)] and the quantal energy are plotted in Figs. 2 and 3 for the specific cases of the points A (Fig. 2) and B (Fig. 3) of Fig. 1. In Fig. 2 we take $\alpha = -1.0025$, $(\delta/2)^2 = 0.01$, and $\Omega = 1.005$ and in Fig. 3 we take $\alpha = -2.7$, $(\delta/2)^2 = 7$, and $\Omega = 2.8284$. Frictionless results ($\delta = 0$) are also given and exhibit a characteristic quasiperiodic behavior. We take $\hbar = 1$, so that the pertinent quantities become dimensionless. The initial conditions are $\langle\hat{x}\rangle = 1$, $\langle\hat{p}\rangle = 1.5$, $s = 10$, $p_s = 15$, $\langle\hat{x}^2\rangle = 3.5$, $\langle\hat{p}^2\rangle = 4.5$, and $\langle\hat{L}\rangle = 1$. Other initial values (like those corresponding to the energies and quantal correlations) can easily be deduced from the given ones.

B. A two-level system coupled to a classical harmonic oscillator

The interaction between a spin or a two-level system with a single mode of the electromagnetic field is a problem of great interest in several fields, e.g., quantum optics, quantum electronics, and magnetic resonance. We consider a two-level Hamiltonian coupled to a classical oscillator (note that dimensionless quantities are employed)

$$\hat{H} = E_1\hat{a}_1^\dagger\hat{a}_1 + E_2\hat{a}_2^\dagger\hat{a}_2 + \frac{\omega}{2}(p_s^2 + s^2) + \gamma s(\epsilon a_1^\dagger\hat{a}_2 + \epsilon^\dagger\hat{a}_2^\dagger\hat{a}_1), \quad (3.27)$$

where we assume $E_2 > E_1$. Here γ is a coupling constant with dimension of energy, ϵ is chosen as a dimensionless parameter, $\hat{a}_1^\dagger, \hat{a}_1$ and $\hat{a}_2^\dagger, \hat{a}_2$ are the creation and the annihilation operators of a particle in, respectively, levels 1 and 2, s is a classical position variable, and p_s is the concomitant momentum. A partial Lie algebra, given by Eqs. (1.1), follows if we choose as relevant operators those belonging to the set $\{\hat{O}_1 = \hat{a}_1^\dagger\hat{a}_1,\ \hat{O}_2 = \hat{a}_2^\dagger\hat{a}_2,\ \hat{O}_3 = i(\epsilon\hat{a}_1^\dagger\hat{a}_2 - \epsilon^\dagger\hat{a}_2^\dagger\hat{a}_1),\ \hat{O}_4 = (\epsilon\hat{a}_1^\dagger\hat{a}_2 + \epsilon^\dagger\hat{a}_2^\dagger\hat{a}_1)\}$. Applying the generalized Ehrenfest theorem to the expectation values of the above mentioned operators we imme-

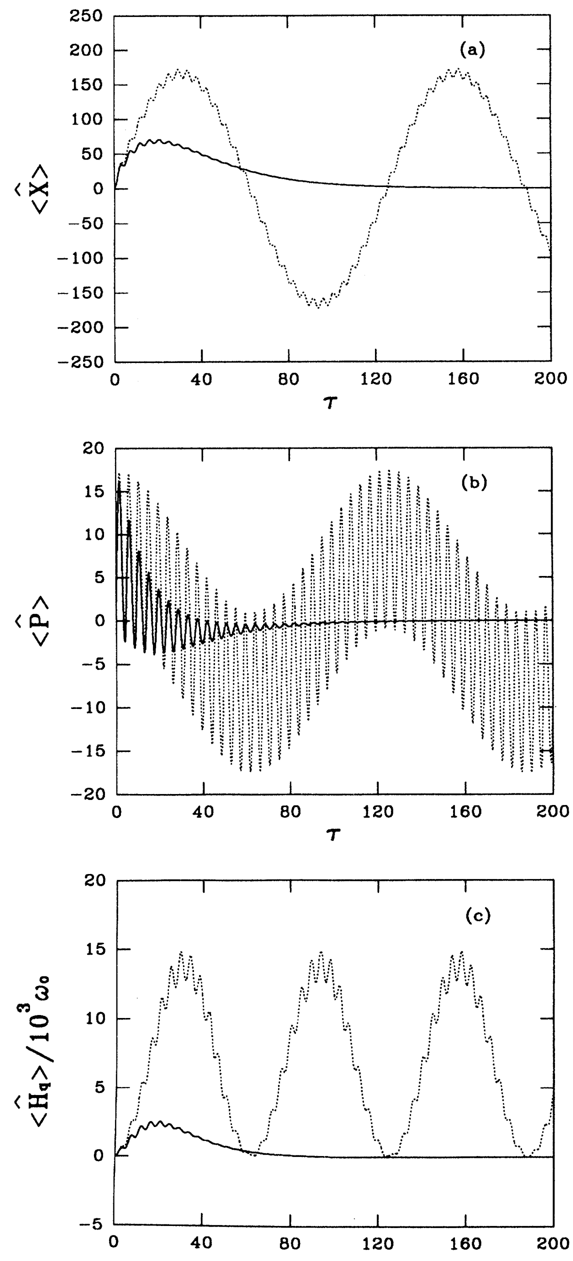

FIG. 2. (a) Temporal evolution of $\langle\hat{x}\rangle$, (b) temporal evolution of $\langle\hat{p}\rangle$, and (c) temporal evolution of quantal energy per unit ω_0. Full line, temporal evolution for point A of Fig. 1 [the values of the parameters are $(\delta/2)^2 = 0.01$, $\alpha = -1.0025$, and $\Omega = 1.005$]; dotted line, temporal evolution for the same value of α and Ω but for $\delta = 0$. As we take $\hbar = 1$, these quantities are dimensionless.

diately arrive at the system of coupled differential equations ($\hbar = 1$)

$$\frac{d\langle\hat{O}_1\rangle}{dt} = -\gamma s\langle\hat{O}_3\rangle, \tag{3.28a}$$

$$\frac{d\langle\hat{O}_2\rangle}{dt} = \gamma s\langle\hat{O}_3\rangle, \tag{3.28b}$$

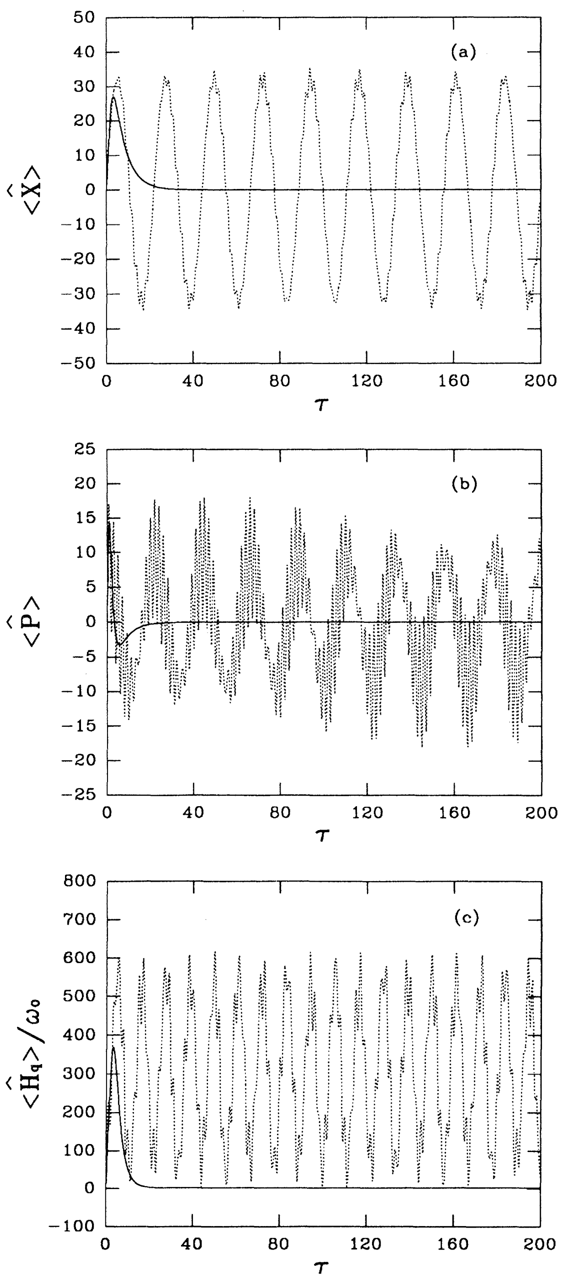

FIG. 3. (a) Temporal evolution of $\langle\hat{x}\rangle$, (b) temporal evolution of $\langle\hat{p}\rangle$, and (c) temporal evolution of quantal energy per unit ω_0. Full line, temporal evolution for point B of Fig. 1 [the values of the parameters are $(\delta/2)^2 = 7$, $\alpha = -2.7$, and $\Omega = 2.8284$]; dotted line, temporal evolution for the same value of α and Ω but for $\delta = 0$. As we take $\hbar = 1$ these quantities are dimensionless.

$$\frac{d\langle\hat{O}_3\rangle}{dt} = -2|\epsilon|^2\gamma s(\langle\hat{O}_2\rangle - \langle\hat{O}_1\rangle) + \omega_0\langle\hat{O}_4\rangle, \tag{3.28c}$$

$$\frac{d\langle\hat{O}_4\rangle}{dt} = -\omega_0\langle\hat{O}_3\rangle, \tag{3.28d}$$

where $\omega_0 = (E_2 - E_1)$. The expectation values $\langle\hat{O}_1\rangle$ and $\langle\hat{O}_2\rangle$ are the populations of levels 1 and 2, respectively, $\langle\hat{O}_3\rangle$ represents a "current" vector, and $\langle\hat{O}_4\rangle$ is the expectation value of the quantal factor of the interaction potential. This system of equations is independent of the nature of the $\hat{a}_1$, $\hat{a}_1^\dagger$, $\hat{a}_2$, and $\hat{a}_2^\dagger$ operators (bosonic or fermionic). For the classical variables we obtain (for the sake of notational simplicity we shall write p instead of p_s), following the usual philosophy [cf. Eq. (2.2b)],

$$\frac{ds}{dt} = \omega p, \tag{3.29a}$$

$$\frac{dp}{dt} = -(\omega s + \gamma\langle\hat{O}_4\rangle + \eta p). \tag{3.29b}$$

We remind the reader that the classical and the quantum variables are dimensionless. We take $|\epsilon| = 1$, for the sake of simplicity. Introducing the population difference operator

$$\Delta\hat{N} = \hat{O}_2 - \hat{O}_1, \tag{3.30}$$

whose mean value is the population difference ΔN and the dimensionless parameter

$$\alpha = \frac{2\gamma}{\omega_0}, \tag{3.31}$$

we obtain the generalized Bloch-like equations [25–27]

$$\frac{d\Delta N}{d\tau} = \alpha s\langle\hat{O}_3\rangle, \tag{3.32a}$$

$$\frac{d\langle\hat{O}_3\rangle}{d\tau} = -\alpha s\Delta N + \langle\hat{O}_4\rangle, \tag{3.32b}$$

$$\frac{d\langle\hat{O}_4\rangle}{d\tau} = -\langle\hat{O}_3\rangle, \tag{3.32c}$$

where $\tau = \omega_0 t$. Further, introducing the dimensionless parameters

$$\Omega = \frac{\omega}{\omega_0}, \qquad \delta = \frac{\eta}{\omega_0}, \tag{3.33}$$

we can recast Eqs. (3.29) in the fashion

$$\frac{ds}{d\tau} = \Omega p, \tag{3.34a}$$

$$\frac{dp}{d\tau} = -(\Omega s + \tfrac{1}{2}\alpha\langle\hat{O}_4\rangle + \delta p). \tag{3.34b}$$

In the five-dimensional space $(\Delta N, \langle\hat{O}_3\rangle, \langle\hat{O}_4\rangle, s, p)$, the fixed points or equilibrium points (denoted with a subindex f) of the nonlinear system of coupled differential equations (3.32) and (3.34) are classified as being of type A or of type B, according to whether the equilibrium value of the classical coordinate s is nonvanishing or equals zero, respectively. By recourse to the invariant of the motion I [21]

$$I = \Delta N^2 + \langle\hat{O}_3\rangle^2 + \langle\hat{O}_4\rangle^2, \tag{3.35}$$

which is the so-called Bloch vector length, these fixed points can be written, for type A, as

$$\Delta N_f = -2\frac{\Omega}{\alpha^2}, \tag{3.36a}$$

$$\langle\hat{O}_3\rangle_f = 0, \tag{3.36b}$$

$$\langle\hat{O}_4\rangle_f = \pm\left(I - 4\frac{\Omega^2}{\alpha^4}\right)^{1/2}, \tag{3.36c}$$

$$s_f = -\frac{\alpha}{2\Omega}\langle\hat{O}_4\rangle_f, \tag{3.36d}$$

$$p_f = 0. \tag{3.36e}$$

The solutions of type A are obtained for $2\Omega/\alpha^2 < I^{1/2}$, where $I^{1/2}$ is the maximum value that $\Delta N(\tau)$ can attain [cf. Eq. (3.35)] for the given initial values $\langle\hat{O}_3\rangle(0)$, $\langle\hat{O}_4\rangle(0)$, and $\Delta N(0)$. Otherwise we have to deal with type B, given by

$$\Delta N_f = \pm I^{1/2}, \tag{3.37a}$$

$$\langle\hat{O}_3\rangle_f = 0, \tag{3.37b}$$

$$\langle\hat{O}_4\rangle_f = 0, \tag{3.37c}$$

$$s_f = 0, \tag{3.37d}$$

$$p_f = 0. \tag{3.37e}$$

It can seen that if the particular case $\Omega = 0$ (corresponding to a classical free particle) is considered, type B is the only possible instance. Notice that the position of the fixed points depends upon the quantal initial conditions only through the value of I.

The stability of the fixed points is determined, in the usual way, by linearizing (3.32) and (3.34) around a fixed point and finding the eigenvalues of the associated matrix (see Appendix B). For type A we obtain either

$$r = 0 \tag{3.38}$$

or, alternatively,

$$r^4 + \delta r^3 + (\Omega^2 + 1 + \alpha^2 s_f^2)r^2 + \delta(1 + \alpha^2 s_f^2)r + \Omega^2\alpha^2 s_f^2 = 0, \tag{3.39}$$

while for type B we find either

$$r = 0 \tag{3.40}$$

or, alternatively,

$$r^4 + \delta r^3 + (\Omega^2 + 1)r^2 + \delta r + \Omega^2 + \alpha^2\Omega\Delta N_f/2 = 0. \tag{3.41}$$

In general, the roots of Eqs. (3.39) and (3.41) can be cataloged in the following fashion: (i) two pairs of complex conjugate roots, (ii) two real roots together with a pair of complex conjugate ones, (iii) four real roots. The real part of the complex conjugate roots does not vanish. Negative real roots and negative real parts of the complex roots are obtained for Eq. (3.39), which entails that the fixed points (3.36) are found to be stable ones (type A).

Negative real roots and negative real parts of the complex roots are obtained for Eq. (3.41) in the range

$$\frac{2\Omega}{\alpha^2} \geq I^{1/2} \tag{3.42}$$

if

$$\Delta N_f = -I^{1/2}. \tag{3.43}$$

If these conditions apply, the fixed points (3.37) (type B) with the minus sign in Eq. (3.37a) are found to be stable. On other hand, if the opposite sign in (3.37a) is chosen, complex roots with positive real parts ensue, so that the fixed points turn out to be unstable ones.

In order to ascertain the attractor character of the stable fixed points, which is here associated with dissipative effects, we have proceeded in the customary fashion by slightly varying both the parameters and the initial conditions. We have, for type A,

$$\Delta N_f = -2\frac{\Omega}{\alpha^2}, \tag{3.44a}$$

$$\langle\hat{O}_4\rangle_f = \pm\left(I - 4\frac{\Omega^2}{\alpha^4}\right)^{1/2}, \tag{3.44b}$$

$$s_f = -\frac{\alpha}{2\Omega}\langle\hat{O}_4\rangle_f, \tag{3.44c}$$

$$p_f = \langle\hat{O}_3\rangle_f = 0, \tag{3.44d}$$

within the range $0 < 2\Omega/\alpha^2 < I^{1/2}$ ("large" coupling), and, for type B,

$$\Delta N_f = -I^{1/2}, \tag{3.45a}$$

$$s_f = \langle\hat{O}_4\rangle_f = 0, \tag{3.45b}$$

whenever we have $2\Omega/\alpha^2 \geq I^{1/2}$.

For both types (A and B) the equilibrium values p_f and $\langle\hat{O}_3\rangle_f$ are zero. Notice that the final particle flux is directed from the excited state towards the ground state. The fixed points depend on the initial conditions (through I) and also on the parameters α and Ω. Notice the abrupt changes that ensue in (3.44) when

$$\frac{2\Omega}{\alpha^2} = I^{1/2}. \tag{3.46}$$

It is worthwhile to point out that the dissipative parameter δ does not influence the nature of the fixed points. In other words, this nature is only affected by the interaction between the classical and the quantum system. The attractors exist because δ is not zero, but their location is independent of the precise value that δ may adopt.

The level populations $\langle\hat{O}_2\rangle$ and $\langle\hat{O}_1\rangle$ may be written in the form

$$\langle\hat{O}_2\rangle(\tau) = \frac{1}{2}[N + \Delta N(\tau)], \tag{3.47a}$$

$$\langle\hat{O}_1\rangle(\tau) = \frac{1}{2}[N - \Delta N(\tau)], \tag{3.47b}$$

where N is the mean value of the total particle number operator $\hat{N} = \hat{O}_1 + \hat{O}_2$, which is an invariant of the motion. Due to the fact that $\Delta N_f \leq 0$, it is found that

$$\langle \hat{O}_2 \rangle_f \leq \langle \hat{O}_1 \rangle_f, \quad (3.48)$$

independently of the initial conditions and the values of parameters. However,

$$\Delta \langle \hat{O}_2 \rangle = \langle \hat{O}_2 \rangle_f - \langle \hat{O}_2 \rangle(0) \quad (3.49)$$

can be positive, for type A, if

$$\frac{2\Omega}{\alpha^2} < -\Delta N(0), \quad (3.50)$$

with $\Delta N(0) < 0$. This case can be used to describe laser excitation: the energy necessary for the transition to the upper level is provided by the classical system, which may represent a single mode of frequency Ω of an electromagnetic field described by the conjugate variables s and p. Part of the energy is dissipated. For type B one finds $\Delta \langle \hat{O}_2 \rangle \leq 0$ for all values of the initial conditions and pertinent parameters. In this case s_f and the potential interaction vanish for $\tau \to \infty$, so that the quantum system and the classical one become decoupled for large times. The expression of the variation of the quantum energy ΔE_q, for

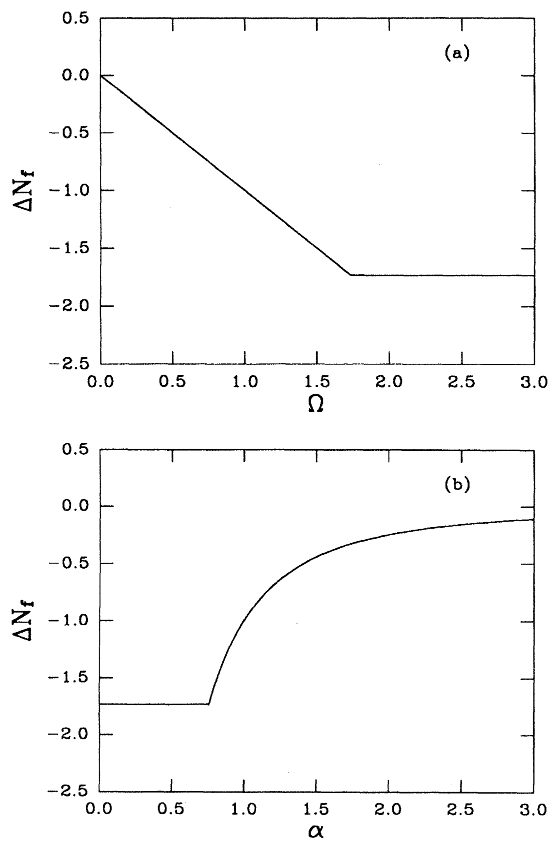

FIG. 4. Behavior of the component corresponding to ΔN of the fixed points of the system of equations (3.32) and (3.34) versus Ω for $\alpha = 2^{1/2}$ and versus α for $\Omega = 0.5$, respectively.

$$\hat{H}_q = E_1 \hat{O}_1 + E_2 \hat{O}_2, \quad (3.51)$$

may be written in terms of ΔN [cf. (3.47)] as

$$\Delta E_q = \langle \hat{H}_q \rangle_f - \langle \hat{H}_q \rangle(0) = -\frac{1}{2}\omega_0 [I^{1/2} + \Delta N(0)] \leq 0, \quad (3.52)$$

where $\langle \hat{H}_q \rangle_f$ is the minimum value of $\langle \hat{H}_q \rangle$ (and also of $\langle \hat{H} \rangle$) for the given initial conditions represented by the

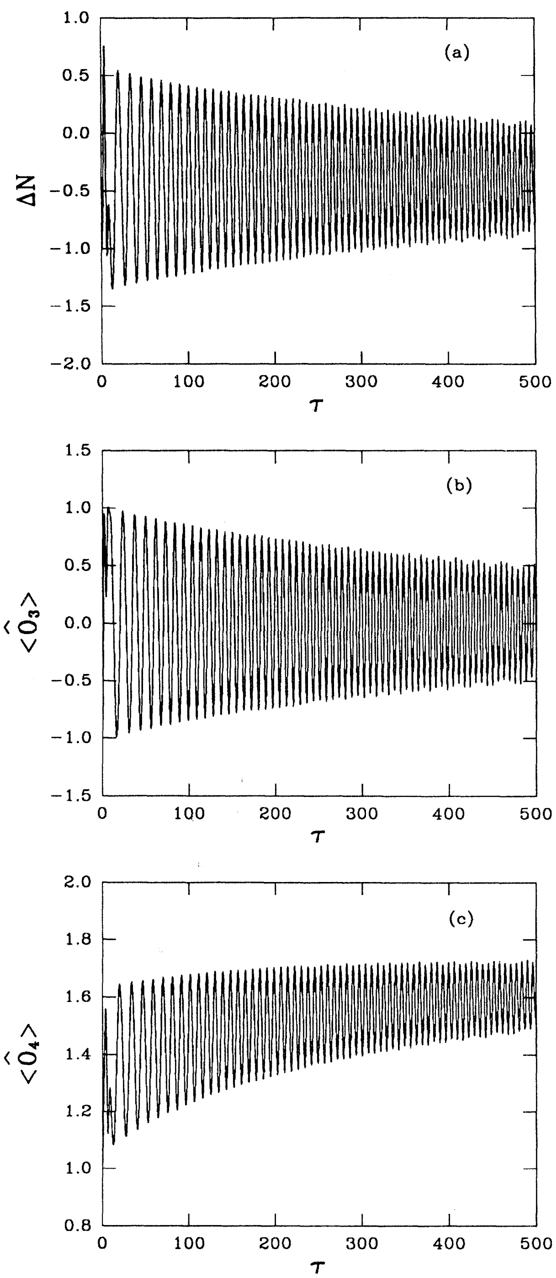

FIG. 5. (a) Temporal evolution of ΔN, (b) temporal evolution of $\langle \hat{O}_3 \rangle$, and (c) temporal evolution of $\langle \hat{O}_4 \rangle$. The values of the parameters are $\alpha = 2$, $\delta = 1$, and $\Omega = 0.8$. The corresponding fixed point is $(-0.4, 0, 1.6852, -2.1065, 0)$.

invariant (3.35).

For type A $s_f \neq 0$ and then the potential interaction does not vanish for $\tau \to \infty$, leading, as in the previous example, to the following final Hamiltonian

$$\hat{H}_f = \frac{1}{2}\omega_0\left(\Delta\hat{N} - \frac{\alpha^2}{\Omega}\langle\hat{O}_4\rangle_f\hat{O}_4\right), \qquad (3.53)$$

where we have not considered the terms $(\omega_0\alpha^2/8\Omega)\langle\hat{O}_4\rangle_f^2$ and $\omega_0\hat{N}/2$, which conmute with the relevants operators. This Hamiltonian describes our coupled system for $\tau \gg 1$. One has

$$\Delta E_q = \langle\hat{H}_f\rangle - \langle\hat{H}_q\rangle(0) = -\frac{1}{2}\omega_0\left(\frac{\alpha^2}{2\Omega}I + \Delta N(0)\right) \leq 0. \qquad (3.54)$$

The sign of ΔE_q is independent of the constant term

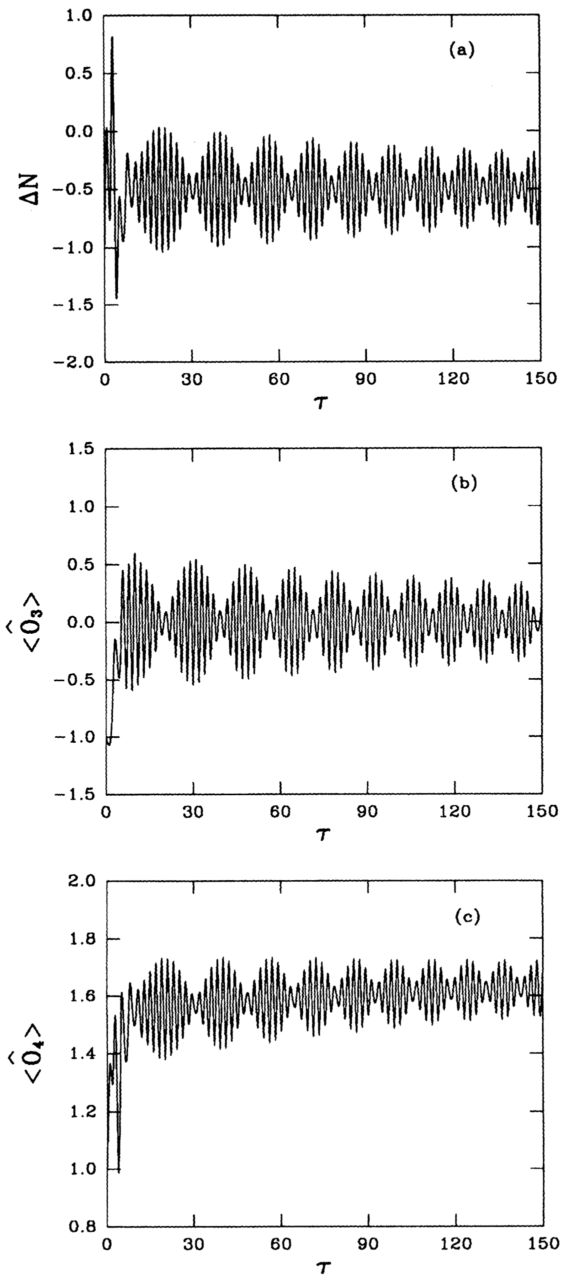

FIG. 6. (a) Temporal evolution of ΔN, (b) temporal evolution of $\langle\hat{O}_3\rangle$, and (c) temporal evolution of $\langle\hat{O}_4\rangle$. The values of the parameters are $\alpha = 2$, $\delta = 1$, and $\Omega = 1$. The corresponding fixed point is $(-0.5, 0, 1.6583, -1.6583, 0)$.

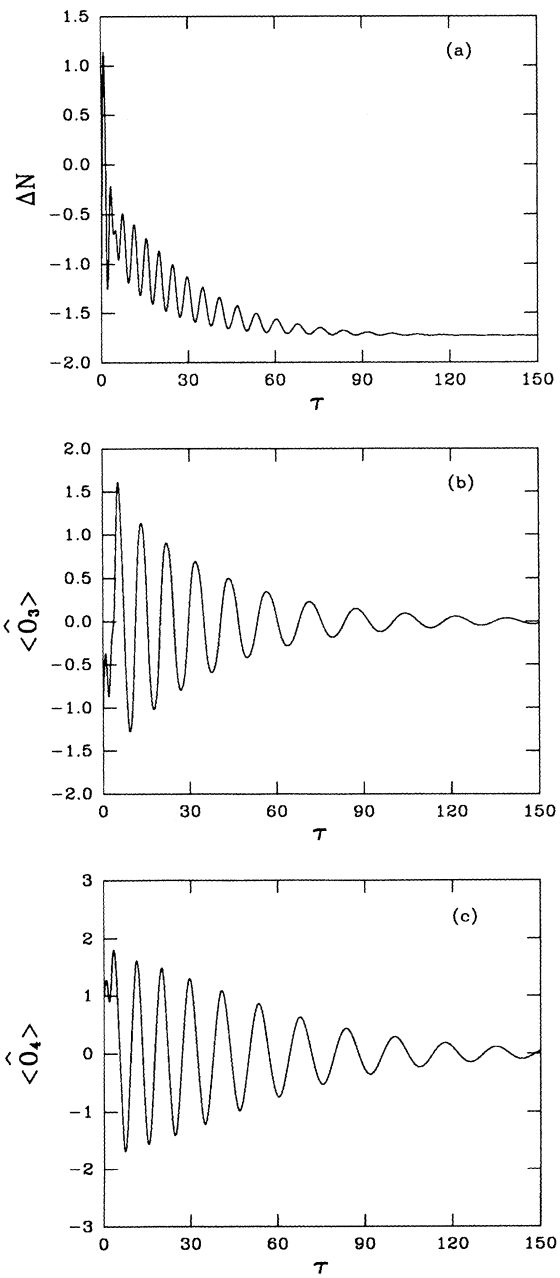

FIG. 7. (a) Temporal evolution of ΔN, (b) temporal evolution of $\langle\hat{O}_3\rangle$, and (c) temporal evolution of $\langle\hat{O}_4\rangle$. The values of the parameters are $\alpha = 2$, $\delta = 1$, and $\Omega = 4$. The corresponding fixed point is $(-1.7321, 0, 0, 0, 0)$.

referred to above.

The Hamiltonian (3.53), obtained in the limiting case, coincides with that of Bonilla and Guinea [4] obtained in the dissipationless case, for $\Omega \gg 1$ [with the equivalence $(\Delta\hat{N}, \hat{O}_3, \hat{O}_4) \leftrightarrow (-\sigma_x, \sigma_y, \sigma_z)$]. The system of equations obtained for the Hamiltonian given by (3.53) is formally the same as that obtained by Bonilla and Guinea [see Eqs. (3.9) of Ref. [4]].

However, since this similitude holds only for $\tau \to \infty$, the nature of the fixed points given here by Eqs. (3.36a)–(3.36c) and (3.37a)–(3.37c) differs from that of [4]. For Bonilla and Guinea Eqs. (3.36a)–(3.36c) and (3.37a)–(3.37c) represent stable centers. For us they represent attractors. The special case of point $(-1,0,0)$ [corresponding to the $(1,0,0)$ point of Bonilla and Guinea] is characterized as an unstable fixed point here and by a saddle point by Bonilla and Guinea.

For the dissipative case of Ref. [4], the fixed points that are obtained in the limit $\alpha^2/\Omega \gg 1$ coincide with ours in the same limit. They are identified as the $(0,0,\pm1)$ points.

Figure 4 depicts the behavior of the component corresponding to ΔN of the stable fixed points of the system of equations (3.32) and (3.34) both versus Ω (for $\alpha = 2^{1/2}$) and versus α (for $\Omega = 0.5$). The temporal evolution of the expectation values of $\Delta\hat{N}$, $\hat{O}_3$, and $\hat{O}_4$ is plotted in Figs. 5, 6 (corresponding to type-A fixed points), and 7 (corresponding to type-B fixed points), for $\alpha = 2$, $\delta = 1$, and $\Omega = 0.8$, 1, and 4, respectively. As we have mentioned before, the dissipative behavior is enhanced for B-type fixed points. We take $\hbar = 1$, so that the pertinent quantities become dimensionless. We consider the bosonic case for $N \geq 3^{1/2}$. The initial conditions are $\Delta N = -1$, $\langle\hat{O}_3\rangle = -1$, $\langle\hat{O}_4\rangle = 1$, $s = 0$, and $p = -10$.

IV. DISCUSSION

Table I summarizes our main results. Two different types of fixed points A and B are found in the present considerations. Fixed points of type A minimize $\langle\hat{H}\rangle$ (for the given initial conditions, as represented by the invariants of the pertinent problem). Fixed points of type B minimize both $\langle\hat{H}\rangle$ and $\langle\hat{H}_q\rangle$. Fixed points B represent a state of maximum relaxation: $\langle\hat{x}\rangle_f = \langle\hat{p}\rangle_f = s_f = p_f = 0$ and a minimum value of the quantum energy for the case of the two interacting oscillators. As regards the two-level system, one encounters for fixed points of type B a situation with $\langle\hat{O}_3\rangle = \langle\hat{O}_4\rangle = s = p_s = 0$ and a maximum number of particles in the lowest-lying level compatible with the initial conditions. The two systems, quantum and classical, interact and then decouple for $\tau \to \infty$. Thus, by looking at the value of the classical variables, one can ascertain the value of the quantum ones. In a loose sense, we can regard the classical system as a sort of "measuring instrument." Some quantum values are correlated to the classical ones and one need look only at the latter.

For type-A fixed points, the quantum and the classical systems become entangled and cannot be separated at $\tau \to \infty$. Even if by looking at the value of the classical variables one can also learn the value of same quantum variables, the situation is not as neat as in type B. On the other hand, these quantum variables do not acquire "most relaxed" values. Notice that, for the two oscillators, we do not look at the values of $\langle\hat{x}^2\rangle_f, \langle\hat{p}^2\rangle_f$, and $\langle\hat{L}\rangle_f$ because they configure a limit cycle.

The idea that could then be advanced is that, by careful calibration of the parameters governing the quantum-classical interaction, one could be in a position to "read" the value of quantum variables just by looking at those of the classical system. Of course, much additional work would be needed along such a line of reasoning.

Summing up, we conclude that by consideration of two coupled systems, a quantum system plus a classical one, we can mimic the dissipating behavior of the quantum system, without recourse to any quantization procedure. The basic commutator $[\hat{a}, \hat{a}^\dagger] = 1$ (or $[\hat{x}, \hat{p}] = i\hbar$), from which the rest of commutation relationships are derived, is satisfied for all t, due to the fact that the dissipating character of the system is introduced via the classical part of the Hamiltonian. For this reason, quantum characteristics are not altered. The quantum dynamical equations reflect dissipation through the s variable, whose temporal evolution is governed by Eqs. (2.2). Fixed points are found, which allows one to speculate on the possibility of using them to determine possible uses of the classical system as a measuring instrument.

TABLE I. Fixed points corresponding to the two examples discussed in this work. The quantity I is given by Eq. (3.35) while $E = (E_1 + E_2)/2\omega_0$.

Hamiltonian (dimensionless)	Condition	Stable fixed points in u space [a]	Type
		$(\langle\hat{x}\rangle_f, \langle\hat{p}\rangle_f, s_f, p_f)$ [b]	
$\frac{\hat{H}}{\omega_0} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) + \frac{\Omega^2}{2}(p_s^2 + s^2)$	$\Omega = \chi^2$	$(-\chi s_f, 0, s_f, 0)$ [c]	A $(s_f \neq 0)$
$+\chi s\hat{x}$	$\Omega > \chi^2$	$(0,0,0,0)$	B $(s_f = 0)$
		$(\Delta\hat{N}_f, \langle\hat{O}_3\rangle_f, \langle\hat{O}_4\rangle, s_f, p_f)$	
$\frac{\hat{H}}{\omega_0} = \frac{1}{2}\Delta\hat{N} + E\hat{N} + \frac{\Omega^2}{2}(p_s^2 + s^2)$	$\Omega < \frac{I^{1/2}}{2}\alpha^2$	$[-\frac{2\Omega}{\alpha^2}, 0, -\frac{2\Omega}{\alpha}s_f, \pm\frac{\alpha}{2\Omega}(I - \frac{4\Omega^2}{\alpha^4})^{1/2}, 0]$	A $(s_f \neq 0)$
$+\alpha s\hat{O}_4$	$\Omega \geq \frac{I^{1/2}}{2}\alpha^2$	$(-I^{1/2}, 0, 0, 0)$	B $(s_f = 0)$

[a] Attractors.

[b] Fixed points of the set of equations (3.3a), (3.3b), and (3.4).

[c] The quantity s_f is given by Eq. (3.13).

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APPENDIX A: QUARTIC EQUATION

Introducing Eq. (3.11) into Eq. (3.8), we obtain

$$v^4 + pv^2 + q = 0 \tag{A1}$$

if the relationship (3.10) holds, where p and r are given by

$$p = \frac{1}{2}\left[4 - \left(\frac{\delta}{2}\right)^2\right], \tag{A2a}$$

$$q = \left[\frac{1}{4}\left(\frac{\delta}{2}\right)^2 + 1\right]^2 - \alpha^2. \tag{A2b}$$

The roots of Eq. (A1) adopt the forms

$$v_1 = (Z^+)^{1/2}, \qquad v_2 = -(Z^+)^{1/2}, \tag{A3a}$$

$$v_3 = (Z^-)^{1/2}, \qquad v_4 = -(Z^-)^{1/2}, \tag{A3b}$$

where

$$Z^+ = -\frac{p}{2} + \Delta^{1/2}, \tag{A4a}$$

$$Z^- = -\left(\frac{p}{2} + \Delta^{1/2}\right), \tag{A4b}$$

with

$$\Delta = \left(\frac{p}{2}\right)^2 - q = -\left(\frac{\delta}{2}\right)^2 + \alpha^2. \tag{A5}$$

Looking at Eqs. (A3) we see that if

$$\alpha^2 < \left(\frac{\delta}{2}\right)^2, \tag{A6}$$

which implies $\Delta < 0$, two pairs of complex conjugate roots are obtained. If instead

$$\alpha^2 \geq \left(\frac{\delta}{2}\right)^2 \tag{A7}$$

($\Delta \geq 0$), two different cases ensue. (a) If $q < 0$ (which is only possible if $\Delta > 0$), i.e.,

$$\alpha^2 > \left[\frac{1}{4}\left(\frac{\delta}{2}\right)^2 + 1\right]^2, \tag{A8}$$

two pure imaginary and two real roots are obtained. (b) If $q \geq 0$, i.e.,

$$\alpha^2 \leq \left[\frac{1}{4}\left(\frac{\delta}{2}\right)^2 + 1\right]^2, \tag{A9}$$

with $p > 0$, i.e.,

$$\left(\frac{\delta}{2}\right)^2 < 4, \tag{A10}$$

two pairs of pure imaginary complex conjugate roots are obtained.

Finally, if $p \leq 0$, i.e.,

$$\left(\frac{\delta}{2}\right)^2 \geq 4, \tag{A11}$$

four real roots are obtained. In the particular case $q = 0$, we have $v_1 = v_2 = 0$. If $q = 0$ and $p = 0$ we obtain $v_1 = v_2 = v_3 = v_4 = 0$. Equations (3.8) and (A1) lead to equivalent roots, although shifted in $\delta/4$,

$$r_i = v_i - \frac{\delta}{4}, \qquad i = 1, 2, \ldots, 4. \tag{A12}$$

These results and the condition [Eq. (3.9)]

$$\alpha^2 \leq \left(\frac{\delta}{2}\right)^2 + 1 \tag{A13}$$

are illustrated in Fig. 1.

APPENDIX B: LINEARIZATION PROCEDURE

The stability of the fixed points is determined by linearizing (3.32) and (3.34) around the fixed points, i.e., we substitute

$$(\Delta N, \langle\hat{O}_3\rangle, \langle\hat{O}_4\rangle, s, p) = (\Delta N_f + \epsilon_{\Delta N}, \langle\hat{O}_3\rangle_f + \epsilon_3, \langle\hat{O}_4\rangle_f + \epsilon_4, s_f + \epsilon_s, p_f + \epsilon_p) \tag{B1}$$

into these equations, which leads, for both types A and B, to the system of equations

$$\frac{d\epsilon_{\Delta N}}{d\tau} = \alpha s_f \epsilon_3, \tag{B2a}$$

$$\frac{d\epsilon_3}{d\tau} = -\alpha(s_f\epsilon_{\Delta N} + \Delta N_f \epsilon_s) + \epsilon_4, \tag{B2b}$$

$$\frac{d\epsilon_4}{d\tau} = -\epsilon_3, \tag{B2c}$$

$$\frac{d\epsilon_s}{d\tau} = \Omega\epsilon_p, \tag{B2d}$$

$$\frac{d\epsilon_p}{d\tau} = -(\Omega\epsilon_s + \tfrac{1}{2}\alpha\epsilon_4 + \delta\epsilon_p). \tag{B2e}$$

For type A the eigenvalues of the concomitant secular matrix are the roots of Eqs. (3.38) and (3.39), while for type B they are the roots of Eqs. (3.40) and (3.41). For type A one obtains either negative real roots or complex real roots with negative real parts as the roots of Eq. (3.39). The solutions of Eqs. (B2) evolve, for $\tau \to \infty$, towards values consistent with a root $r = 0$ [cf. Eq. (3.38)],

$$\epsilon_{\Delta N_f} = \epsilon_{3_f} = \epsilon_{p_f} = 0, \tag{B3a}$$

$$\epsilon_{s_f} = -\frac{\alpha}{2\Omega}\epsilon_{4_f}. \tag{B3b}$$

The value of ϵ_{4_f} in Eq. (B3b) is determined using the invariant of the motion $I(\epsilon)$ corresponding to Eqs. (B2)

$$\epsilon_{4_f} = \frac{I(\epsilon)}{2\langle\hat{O}_4\rangle_f}, \tag{B4}$$

where $I(\epsilon)$ is given by

$$I(\epsilon) = 2[\Delta N_f \epsilon_{\Delta N}(0) + \langle\hat{O}_4\rangle_f \epsilon_4(0)]. \tag{B5}$$

Therefore, trajectories starting near the fixed points (3.36) move toward neighboring fixed points (denoted with a superindex n), given by

$$\Delta N_f^n = \Delta N_f, \tag{B6a}$$

$$\langle\hat{O}_4\rangle_f^n = \langle\hat{O}_4\rangle_f + \frac{I(\epsilon)}{2\langle\hat{O}_4\rangle_f}, \tag{B6b}$$

$$s_f^n = s_f - \frac{\alpha}{4\Omega}\frac{I(\epsilon)}{\langle\hat{O}_4\rangle_f}, \tag{B6c}$$

$$p_f^n = p_f, \tag{B6d}$$

$$\langle\hat{O}_3\rangle_f^n = \langle\hat{O}_3\rangle_f. \tag{B6e}$$

Negative real roots and negative real parts of the complex roots are obtained for Eq. (3.41) if, in Eq. (3.37a),

$$\Delta N_f = -I^{1/2}, \tag{B7}$$

and the relation (3.42) is satisfied. As a consequence, the solutions of Eqs. (B2) evolve, for $\tau \to \infty$, to values corresponding to the eigenvalue zero [Eq. (3.40)], as in type A. These constants [the fixed points of Eqs. (B2)] are determined by Eqs. (B2) and (B5) and one finds

$$\epsilon_{\Delta N_f} = \frac{I(\epsilon)}{2\Delta N_f}, \tag{B8a}$$

$$\epsilon_{3_f} = \epsilon_{4_f} = \epsilon_{s_f} = \epsilon_{p_f} = 0, \tag{B8b}$$

which determine the fixed points

$$\Delta N_f^n = \Delta N_f + \frac{I(\epsilon)}{2\Delta N_f}, \tag{B9a}$$

$$\langle\hat{O}_4\rangle_f^n = \langle\hat{O}_4\rangle_f, \tag{B9b}$$

$$s_f^n = s_f, \tag{B9c}$$

$$p_f^n = p_f, \tag{B9d}$$

$$\langle\hat{O}_3\rangle_f^n = \langle\hat{O}_3\rangle_f. \tag{B9e}$$

In general, in both cases, the trajectories starting in the vicinity of the fixed points move toward neighboring fixed points. The concomitant shift, as it is easy to see [cf. Eqs. (B6) and (B9)], is proportional to the perturbation. In particular, if the perturbation does not modify the invariant I, i.e.,

$$[\Delta N_f + \epsilon_{\Delta N}(0)]^2 + [\langle\hat{O}_3\rangle_f + \epsilon_3(0)]^2 + [\langle\hat{O}_4\rangle_f + \epsilon_4(0)]^2 \simeq I + I(\epsilon) = I, \tag{B10}$$

so that $I(\epsilon) = 0$ (we have neglected powers of ϵ higher than the linear one; note that in both cases $\langle\hat{O}_3\rangle_f$ vanishes), the trajectories lead back to the original fixed points. Therefore, if we start in the neighborhood of the fixed points, we either return to them or remain in their vicinity. The fixed points are stable ones [26]. On the other hand, if the opposite sign in (3.37a) is chosen, complex roots with positive real parts ensue, so that the fixed points turn out to be unstable ones.

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